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TRAINING AND DEVELOPMENT OF CREATIVE CAPACITY OF STUDENTS IN MATHEMATICS
- Examples of good practices –

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Abstract

The school reality presents us with two negative things, for the vast majority of students, in each class: on the one hand, Mathematics is a dry school subject, difficult to penetrate and, on the other hand, they learn Mathematics, thanks to one or more external motivations. In other words, Mathematics is a hard subject that is learned by force, not for pleasure. On the other hand, it is known that in order to learn Mathematics successfully, you must have a certain level of development of logical-mathematical thinking, but also a level of creative ability. Of course, these two levels increase as you learn Mathematics. In other words, the formation and development of logical-mathematical thinking and the ability to create is the cause and effect of learning Mathematics. Therefore, the Mathematics teacher must always have in mind, for each student, the raising of these two levels. In this paper we will present a concrete way to achieve this fact, "*playing*" with three equilateral triangles, in the idea of training and developing the competences to solve such problems. Thus, we will consider three equilateral triangles of different sides, each of which has one side located on a straight line d and the other sides located on the same side of this straight line. Moreover, the sides of these triangles which lie on the same straight line are in extension. We will determine, in this paper, a series of metric relations between the sides of these triangles, so that the angle formed by the three vertices not located on the right d is of an arbitrary measure. At the end of the paper, I proposed to the reader attentive and interested in these issues, the solution of four complementary problems to those solved in the paper.

Keywords

Intelligence, Creativity, Mathematics, Angle, Equilateral Triangle

1. Introduction

As I stated above, the school reality presents us with two negative things, for the vast majority of students, in every class: on the one hand, Mathematics is a dry school subject, difficult to penetrate, and, on the other hand, they learn Mathematics, due to one or some external motivations. In other words, Mathematics is a hard subject that is learned forcibly, for fear of a low grade, not for pleasure.

On the other hand, it is known that in order to learn Mathematics successfully, you must have a certain level of development of logical-mathematical thinking, but also a level of creative ability. Of course, these two levels increase as you learn Mathematics. In other words, the formation and development of logical-mathematical thinking and the ability to create is the cause and effect of learning Mathematics.

Therefore, the Mathematics teacher must always have in mind, for each student, the raising of these two levels. Of course, to do this, he must have very good logical-mathematical thinking and a high level of creative ability; that is, to be intelligent and very creative.

In this paper we will present concrete ways to achieve this fact, "*playing*" with three equilateral triangles. At the end of the paper, I proposed to the attentive reader interested in these issues, four problems to solve.

We specify that, where there is no danger of confusion, we will use the same notation for the angle and its measure.

2. Main results

Consider four collinear points A, B, C and D, with:

AB=a, BC=b and CD=c.

We also consider the equilateral triangles ABE, BCF and CDG, located on the same side of the line AD. Depending on the order relationship between the numbers a, b and c, we will determine certain necessary and / or sufficient conditions for the angle EFG to have a certain value.

Regardless, however, of the order relation that exists on the set {a,b,c}, and, according to the hypothesis and the Cosine Theorem, applied to the triangles EBF, FCG and EFG, we have the equalities:

$EF^2=a^2+b^2-a\cdot b$ and $FG^2=b^2+c^2-b\cdot c$, (1)

and $EG^2=a^2+2\cdot b^2+c^2-a\cdot b-b\cdot c-2\cdot\sqrt{a^2-a\cdot b+b^2}\cdot\sqrt{b^2-b\cdot c+c^2}\cdot\cos\theta$, (2)

where:

$\theta=\sphericalangle EFG$. (3)

Since there are six order relations on the set {a,b,c}, in our study we will distinguish as many cases. But for reasons of symmetry, these six cases will be reduced to three.

Case 1: c<a<b. In this case, see Figure 1, below.

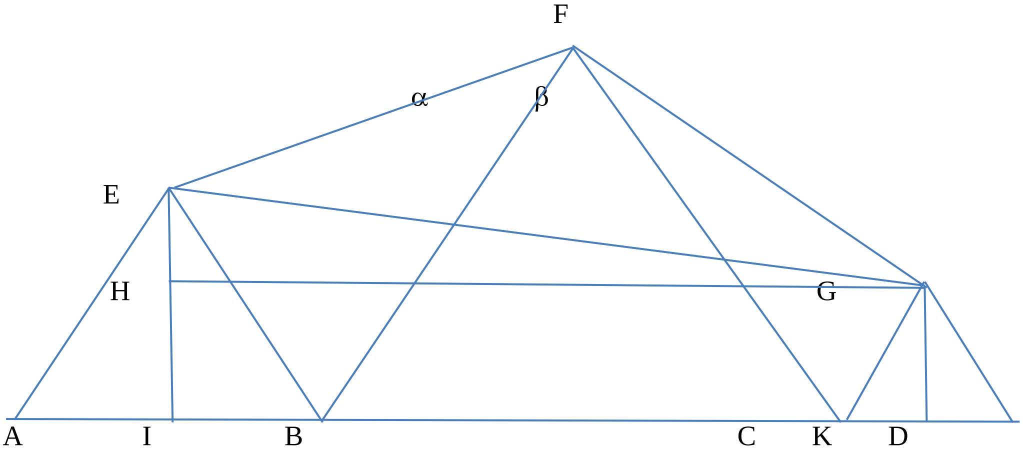

Figure 1.

Indeed, let be EI ⊥ AB, GK ⊥ CD and GH ⊥ EI. Then IKHG is a rectangle:

$GH=b+\dfrac{a+c}{2}$ and $EH=\dfrac{\sqrt{3}}{2}\cdot(a-c)$. (4)

So, according to the Pythagorean Theorem, applied to the right triangle EHG, we obtain:

$$EG^2=\left(b+\frac{a+c}{2}\right)^2+\frac{3}{4}\cdot(a-c)^2. \quad (5)$$

From the equalities (2) and (5), it follows that:

$$a^2+2\cdot b^2+c^2-a\cdot b-b\cdot c-2\cdot\sqrt{a^2-a\cdot b+b^2}\cdot\sqrt{b^2-b\cdot c+c^2}\cdot\cos\theta=\left(b+\frac{a+c}{2}\right)^2+\frac{3}{4}\cdot(a-c)^2, \quad (6)$$

which is equivalent to:

$4\cdot(a^2-a\cdot b+b^2)\cdot(b^2-b\cdot c+c^2)\cdot\cos^2\theta=[b^2-2\cdot b\cdot(a+c)+a\cdot c]^2$. (6′)

From equality (6′) it follows that:

$\theta=90°$ if and only if $b^2-2\cdot b\cdot(a+c)+a\cdot c=0$. (7)

Equation from (7), has the solutions:

$b=a+c-\sqrt{a^2+a\cdot c+c^2}$ and $b=a+c+\sqrt{a^2+a\cdot c+c^2}$. (7′)

For our case, c<a<b, the second equality is valid from (7′).

***Remark 1*:** Equality (7) is also obtained if a<c<b, according to Figure 2:

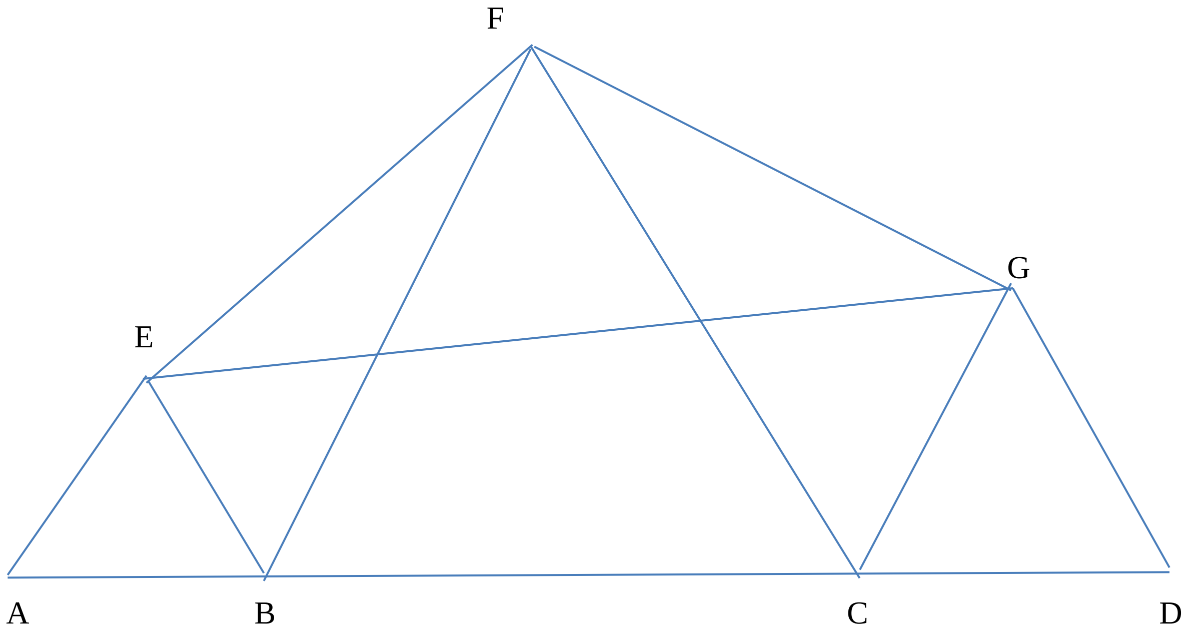

Figure 2.

Returning to Figure 1, from the Cosine Theorem, applied to triangles EFB and CFG, we obtain that:

$$\cos\alpha=\frac{2\cdot b-a}{2\cdot\sqrt{a^2-a\cdot b+b^2}} \quad \text{and} \quad \cos\beta=\frac{2\cdot b-c}{2\cdot\sqrt{c^2-c\cdot b+b^2}}. \quad (8)$$

Now, from the Theorem of Sines, applied to the same triangles EFB and CFG, we obtain that:

$$\sin\alpha=\frac{\sqrt{3}\cdot a}{2\cdot\sqrt{a^2-a\cdot b+b^2}} \quad \text{and} \quad \sin\beta=\frac{\sqrt{3}\cdot c}{2\cdot\sqrt{c^2-c\cdot b+b^2}}. \quad (9)$$

Of course that:

$\theta=90°$ if and only if $\alpha+\beta=30°$.

So:

$\theta=90°$ if and only if $\sin(\alpha+\beta)=\frac{1}{2}$,

which is equivalent to:

$\sin\alpha\cdot\cos\beta+\sin\beta\cdot\cos\alpha=\frac{1}{2}$,

that is:

$$\frac{\sqrt{3}\cdot a}{2\cdot\sqrt{a^2-a\cdot b+b^2}}\cdot\frac{2\cdot b-c}{2\cdot\sqrt{c^2-c\cdot b+b^2}}+\frac{\sqrt{3}\cdot c}{2\cdot\sqrt{c^2-c\cdot b+b^2}}\cdot\frac{2\cdot b-a}{2\cdot\sqrt{a^2-a\cdot b+b^2}}=\frac{1}{2}. \quad (10)$$

Equality (10) is equivalent to:

$$[b^2-2\cdot b\cdot(a+c)+a\cdot c]\cdot[b^2+b\cdot(a+c)+2\cdot a\cdot c]=0. \quad (11)$$

Therefore, we obtained, also in this way, the equivalence (7).

***Remark 2*:** We have the equivalence:

$\theta=90°$ if and only if $EG^2=EF^2+FG^2$. (12)

According to equalities (2) and (5), the metric equality in (12) becomes:

$$\left(b+\frac{a+c}{2}\right)^2+\frac{3}{4}\cdot(a-c)^2=a^2+b^2-a\cdot b+b^2+c^2-b\cdot c,$$

equality which is equivalent to equality (7′).

If θ=120°, then equality (6′) becomes:

$(a^2-a\cdot b+b^2)\cdot(b^2-b\cdot c+c^2)=(b^2-2\cdot a\cdot b-2\cdot b\cdot c+a\cdot c)^2$. (12)

But, equality (12) is equivalent to:

$b^4-b^3\cdot(a+c)+b^2\cdot(a^2+a\cdot c+c^2)-b\cdot(a^2\cdot c+a\cdot c^2)+a^2\cdot c^2=$
$b^4-4\cdot b^3\cdot(a+c)+b^2\cdot(4\cdot a^2+10\cdot a\cdot c+4\cdot c^2)-4\cdot b\cdot(a^2\cdot c+a\cdot c^2)+a^2\cdot c^2$,

that is:

$3\cdot b^3\cdot(a+c)-b^2\cdot(3\cdot a^2+9\cdot a\cdot c+3\cdot c^2)+3\cdot b\cdot(a^2\cdot c+a\cdot c^2)=0$. (13)

From equality (13), it follows that:

$b^2\cdot(a+c)-b\cdot(a^2+3\cdot a\cdot c+c^2)+(a^2\cdot c+a\cdot c^2)=0$. (13′)

Solving equation (13′), we obtain that:

$b=a+c$ or else $b=\frac{a\cdot c}{a+c}$. (14)

The second equality in (14) cannot take place.

So, we can say that, in this case:

θ=120° implies b=a+c. (15)

Now, suppose that b=a+c. Then, the equality (6′) becomes:

$4\cdot[a^2-a\cdot(a+c)+(a+c)^2]\cdot[(a+c)^2-(a+c)\cdot c+c^2]\cdot\cos^2\theta=[(a+c)^2-2\cdot(a+c)^2+a\cdot c]^2$,

that is:

$4\cdot(a^2+a\cdot c+c^2)^2\cdot\cos^2\theta=(a^2+a\cdot c+c^2)^2$;

whence it follows that:

$\cos^2\theta=\frac{1}{4}$ which, by hypothesis, is equivalent to: θ=120°

So implication (15) is equivalence. Therefore, we can say that, in this case:

θ=120° is equivalent to b=a+c. (15′)

***Remark 3*:** We assume that θ=120°. Consider a straight line FT, parallel to AB (see Figure 3) and let be:

BE∩FT={R}.

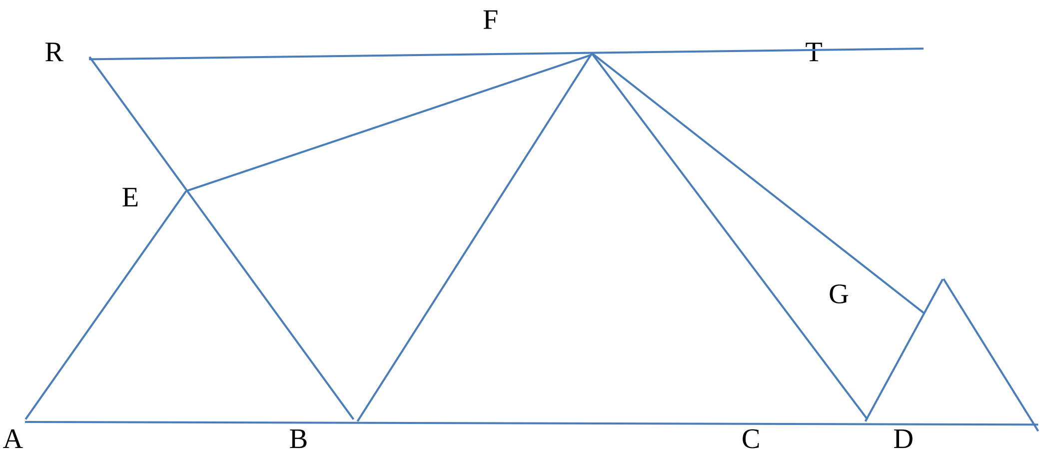

Figure 3.

Then the triangle BFR is equilateral and, according to the hypothesis,

m(∢RFC)=m(∢EFG)=m(∢θ)=120°. (16)

Therefore,

m(∢RFE)=m(∢CFG). (17)

From the hypothesis and equality (16), it follows that the quadrilateral BCFR is a rhombus; so:
RF=FC=b. (18)

It follows, from here, that:
ΔRFE ≡ ΔCFG. (ASA) (19)

According to relation (19), we have the equalities:
From equalities (20), it follows that:
RE=c and EF=FG. (20)

Now, using equality (18) or equalities (1), we get equality from (15′).

Remark 4**:** Let be, ∢CFG=γ. Then:
∢BFE=60°-γ, ∢FEB=60°+γ and ∢FGC=120°-γ.

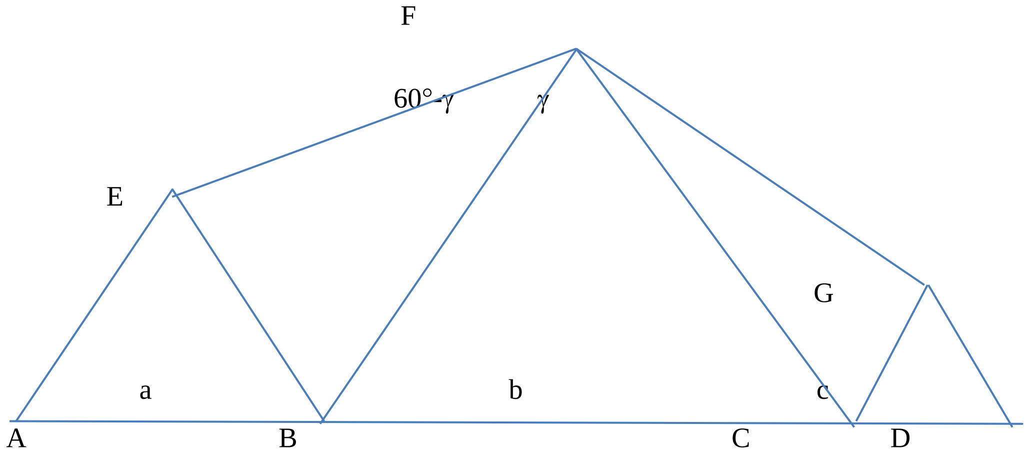

Figure 4.

Then, from the Theorem of Sines applied to the triangles BEF and CFG, we obtain that:

$$\frac{a}{b}=\frac{\sin(60^\circ-\gamma)}{\sin(60^\circ+\gamma)} \quad \text{and} \quad \frac{c}{b}=\frac{\sin\gamma}{\sin(120^\circ-\gamma)}. \tag{21}$$

From equalities (21), it follows that:

$$\frac{a+c}{b}=\frac{\sin(60^\circ-\gamma)}{\sin(60^\circ+\gamma)}+\frac{\sin\gamma}{\sin(120^\circ-\gamma)}=\frac{\sin(60^\circ-\gamma)}{\sin(60^\circ+\gamma)}+\frac{\sin\gamma}{\sin(60^\circ+\gamma)}=\frac{\sin(60^\circ-\gamma)+\sin\gamma}{\sin(60^\circ+\gamma)}$$

$$=\frac{2\cdot\sin 30^\circ\cdot\cos(30^\circ-\gamma)}{\sin(60^\circ+\gamma)}=\frac{2\cdot\frac{1}{2}\cdot\cos(30^\circ-\gamma)}{\sin(60^\circ+\gamma)}=\frac{\cos(30^\circ-\gamma)}{\sin(60^\circ+\gamma)}=\frac{\sin(60^\circ+\gamma)}{\sin(60^\circ+\gamma)}=1. \tag{22}$$

Now, from the extreme equalities in (22), we obtain the equality in (15′).
Next, also for this case, we propose to calculate tgθ.
So, we consider, here too, the notations from Figure 1:
m(∢EFG)=θ, m(∢EFB)=α and m(∢CFG)=β.
Then,
α+β+60°=θ (23)
and, from equality (6′), it follows that:

$$\cos^2\theta=\frac{\left[b^2-2\cdot b(a+c)+a\cdot c\right]^2}{4\cdot(a^2-a\cdot b+b^2)\cdot(c^2-c\cdot b+b^2)} \tag{24}$$

Then,

$$tg^2\theta=\frac{1}{\cos^2\theta}-1=\frac{4\cdot(a^2-a\cdot b+b^2)\cdot(c^2-c\cdot b+b^2)-\left[b^2-2\cdot b(a+c)+a\cdot c\right]^2}{\left[b^2-2\cdot b(a+c)+a\cdot c\right]^2}=\frac{3\cdot(b^2-a\cdot c)^2}{\left[b^2-2\cdot b(a+c)+a\cdot c\right]^2} \quad (25)$$

From equality (25) we obtain the following equivalence:

$\theta=120°$ if and only if $[b^2-2\cdot b\cdot(a+c)+a\cdot c]^2=(b^2-a\cdot c)^2$. (26)

Now, from the last equality in (26), we obtain the equalities in (14).

***Remark 5*:** Using equations (8) and (9), we obtain that:

$$tg\alpha=\frac{\sin\alpha}{\cos\alpha}=\frac{\sqrt{3}\cdot a}{2\cdot b-a} \quad \text{and} \quad tg\beta=\frac{\sin\beta}{\cos\beta}=\frac{\sqrt{3}\cdot c}{2\cdot b-c}. \quad (27)$$

Now, using equalities (27) and the formula:

$$tg\theta=tg(\alpha+\beta+60°)=\frac{tg\alpha+tg\beta+tg60^\circ-tg\alpha\cdot tg\beta\cdot tg60^\circ}{1-tg\alpha\cdot tg\beta-tg\alpha\cdot tg60^\circ-tg\beta\cdot tg60^\circ}, \quad (28)$$

we obtain that:

$$tg\theta=\frac{\sqrt{3}\cdot(b^2-a\cdot c)}{b^2-2\cdot b\cdot(a+c)+a\cdot c}. \quad (25')$$

***Remark 6*:** Consider Figure 5 below,

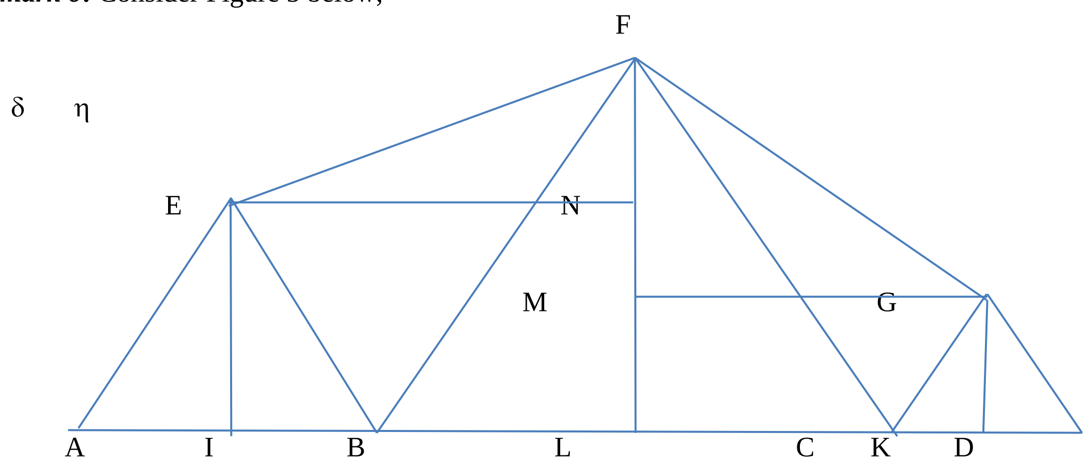

Figure 5.

where:

$EI\perp AB$, $FL\perp BC$, $GK\perp CD$, $EN\perp FL$ and $GM\perp FL$. (29)

If $m(\sphericalangle EFG)=\theta$, $m(\sphericalangle EFN)=\delta$ and $m(\sphericalangle GFM)=\eta$, then,

$\delta+\eta=\theta$ (30)

and

$$EN=\frac{a+b}{2}, \quad FN=\frac{\sqrt{3}\cdot(b-a)}{2}, \quad GM=\frac{b+c}{2} \quad \text{and} \quad FM=\frac{\sqrt{3}\cdot(b-c)}{2}, \quad (31)$$

$$tg\theta=tg(\delta+\eta)=\frac{tg\delta+tg\eta}{1-tg\delta\cdot tg\eta}. \quad (32)$$

But,

$$tg\delta=\frac{EN}{FN}=\frac{a+b}{\sqrt{3}\cdot(b-a)} \quad \text{and} \quad tg\eta=\frac{GM}{FM}=\frac{c+b}{\sqrt{3}\cdot(b-c)}. \quad (33)$$

Now, from equalities (31) and (32), we obtain the equalitiy (25′).

***Remark 7*:** Consider Figure 6, below:

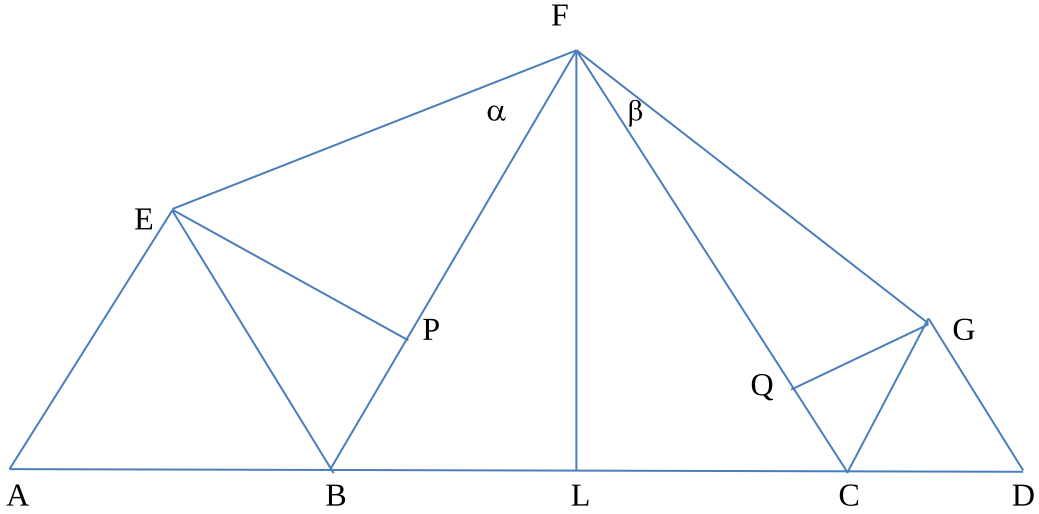

Figure 6.

where:

EP ⊥ FB and GQ ⊥ FC. (34)

If we use the notations in Figure 1:

m(∢EFG)=θ, m(∢EFB)=α and m(∢GFC)=β,

then, again, the next equality holds:

α+β+60°=θ. (23)

Then, in the right triangles EPB and CQG, we have the equalities:

EP=$\frac{a\cdot\sqrt{3}}{2}$, BP=$\frac{a}{2}$ and CQ=$\frac{c\cdot\sqrt{3}}{2}$, CQ=$\frac{c}{2}$. (35)

It follows that:

FP=$\frac{2\cdot b-a}{2}$ and FQ=$\frac{2\cdot b-c}{2}$. (36)

According to the hypothesis, equalities (36) and Figure 6, again, we obtain that:

tgα=$\frac{EP}{FP}=\frac{a\cdot\sqrt{3}}{2\cdot b-a}$ and tgβ=$\frac{EQ}{FQ}=\frac{c\cdot\sqrt{3}}{2\cdot b-c}$. (27)

Now, as above, using equalities (27) and formula (28), we obtain, again the equality (25′).

If θ=135°, then equality (6′) becomes:

$2\cdot(a^2-a\cdot b+b^2)\cdot(b^2-b\cdot c+c^2)=[b^2-2\cdot b\cdot(a+c)+a\cdot c]^2$, (37)

that is:

$[(\sqrt{3}+1)\cdot b^2-2\cdot b\cdot(a+c)-a\cdot c\cdot(\sqrt{3}-1)]\cdot[(\sqrt{3}-1)\cdot b^2+2\cdot b\cdot(a+c)-a\cdot c\cdot(\sqrt{3}+1)]=0$. (37′)

If:

$(\sqrt{3}+1)\cdot b^2-2\cdot b\cdot(a+c)-a\cdot c\cdot(\sqrt{3}-1)=0$, (38)

then:

b=$\frac{\sqrt{3}-1}{2}\cdot(a+c-\sqrt{a^2+4\cdot a\cdot c+c^2})<0$ or else b=$\frac{\sqrt{3}-1}{2}\cdot(a+c+\sqrt{a^2+4\cdot a\cdot c+c^2})$. (39)

If:

$(\sqrt{3}-1)\cdot b^2+2\cdot b\cdot(a+c)-a\cdot c\cdot(\sqrt{3}+1)=0$, (40)

then:

b=$\frac{\sqrt{3}+1}{2}\cdot(-a-c-\sqrt{a^2+4\cdot a\cdot c+c^2})<0$ or else b=$\frac{\sqrt{3}+1}{2}\cdot(-a-c+\sqrt{a^2+4\cdot a\cdot c+c^2})$. (41)

Since c<a<b, it follows that, in this case, equality (37′) is equivalent to equality (38). Therefore, in this

case, the second equality from (39) holds:

$b=\frac{\sqrt{3}-1}{2}\cdot(a+c+\sqrt{a^2+4\cdot a\cdot c+c^2})$ and $c<a<\frac{\sqrt{3}-1}{2}\cdot(a+c+\sqrt{a^2+4\cdot a\cdot c+c^2})$. (42)

From the inequalities in (42), we obtain that if c is fixed, then:

$a\in(c,(2+\sqrt{3})\cdot c)$ and $b\in\left(\frac{(1+\sqrt{3})\cdot(2+\sqrt{6})}{2}\cdot c,\frac{(1+\sqrt{3})\cdot(5+3\cdot\sqrt{3})}{2}\cdot c\right)$. (43)

Remark 8**:** If θ=135°, then α+β=75°. But, then:

$tg(\alpha+\beta)=2+\sqrt{3}$. (44)

Now, using equalities (27) and (44), we obtain the equality:

$\frac{\frac{a\cdot\sqrt{3}}{2\cdot b-a}+\frac{c\cdot\sqrt{3}}{2\cdot b-c}}{1-\frac{a\cdot\sqrt{3}}{2\cdot b-a}\cdot\frac{c\cdot\sqrt{3}}{2\cdot b-c}}=2+\sqrt{3}$, i.e.: $(\sqrt{3}+1)\cdot b^2-2\cdot b\cdot(a+c)-a\cdot c\cdot(\sqrt{3}-1)$. (38)

If θ=150°, then equality (6′) becomes:

$3\cdot(a^2-a\cdot b+b^2)\cdot(b^2-b\cdot c+c^2)=[b^2-2\cdot b\cdot(a+c)+a\cdot c]^2$, (45)

that is:

$[2\cdot b^2-b\cdot(a+c)-a\cdot c]\cdot[b^2+b\cdot(a+c)-2\cdot a\cdot c]=0$. (45′)

Because c<a<b, it follows that $b^2+b\cdot(a+c)-2\cdot a\cdot c>0$. So equality (45′) is equivalent to:

$2\cdot b^2-b\cdot(a+c)-a\cdot c=0$, i.e.: $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{2\cdot b}{a\cdot c}$. (45″)

From the equations from (45″) the solutions result:

$b=\frac{a+c-\sqrt{a^2+10\cdot a\cdot c+c^2}}{4}$ and $b=\frac{a+c+\sqrt{a^2+10\cdot a\cdot c+c^2}}{4}$. (46)

Because the first equality from (46) cannot take place, and c<a<b, from the second equality from (46), it follows that:

$c<a<\frac{a+c+\sqrt{a^2+10\cdot a\cdot c+c^2}}{4}$. (47)

From the inequalities in (47), we obtain that, if c is fixed, then:

$a\in(c,2\cdot c)$ and $b\in\left(\frac{(1+\sqrt{3})}{2}\cdot c,2\cdot c\right)$. (48)

Remark 9**:** If θ=150°, then α+β=90°. But, then:

$tg\alpha=tg(90°-\beta)=ctg\beta=\frac{1}{tg\beta}$, i.e.: $tg\alpha\cdot tg\beta=1$. (49)

Now, using equalities (27), the second equality in (49) becomes:

$\frac{a\cdot\sqrt{3}}{2\cdot b-a}\cdot\frac{c\cdot\sqrt{3}}{2\cdot b-c}=1$, i.e.: $2\cdot b^2-b\cdot(a+c)-a\cdot c=0$. (45″)

If:

θ=150°, then: $tg\theta=-\sqrt{3}$. (50)

Now, from equalities (32), (33) and (50), we obtain the equality (45″).

***Remark 10*:** The equality (45″) also obtains if θ=150° and a<b<c.

Indeed, in this case we use Figure 7:

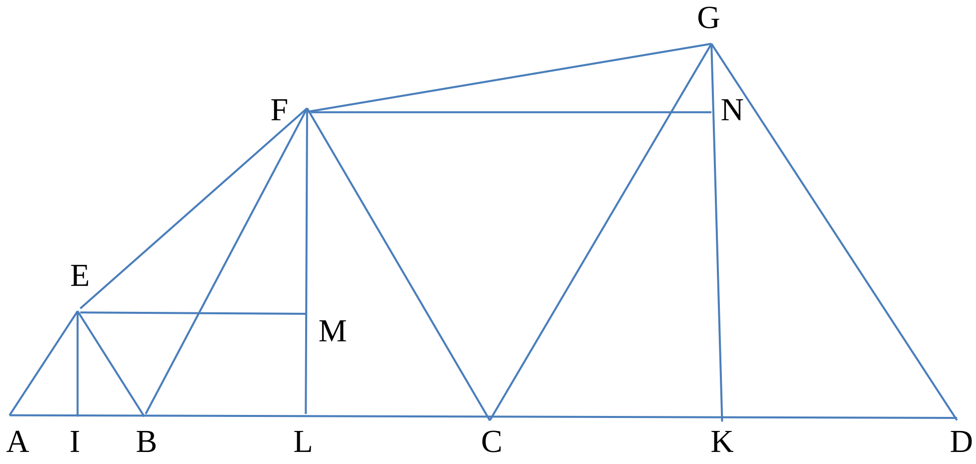

Figure 7.

where:

$$EI \perp AB, \quad FL \perp BC, \quad GK \perp CD, \quad EM \perp FL \quad \text{and} \quad GN \perp FL. \tag{51}$$

If m(∢EFG)=θ, m(∢EFM)=λ and m(∢GFN)=μ, then,

$$\lambda+\mu+90°=\theta=150° \tag{52}$$

and

$$EM=\frac{a+b}{2}, \quad FM=\frac{\sqrt{3}\cdot(b-a)}{2}, \quad GN=\frac{b+c}{2}, \quad FN=\frac{\sqrt{3}\cdot(b-c)}{2}. \tag{53}$$

But, from equalities (53), we obtain that:

$$tg\lambda=\frac{EM}{FM}=\frac{a+b}{\sqrt{3}\cdot(b-a)} \quad \text{and} \quad tg\mu=\frac{GN}{FN}=\frac{c+b}{\sqrt{3}\cdot(b-c)}. \tag{54}$$

Now, from equalities (52) and (54), we obtain that:

$$\sqrt{3}=tg60°=tg(\delta+\eta)=\frac{tg\delta+tg\eta}{1-tg\delta\cdot tg\eta}=\frac{\frac{a+b}{\sqrt{3}\cdot(b-a)}+\frac{c+b}{\sqrt{3}\cdot(b-c)}}{1-\frac{a+b}{\sqrt{3}\cdot(b-a)}\cdot\frac{c+b}{\sqrt{3}\cdot(b-c)}}. \tag{55}$$

Finally, from the equalities in (55), we obtain the equality (45″). So the following equality holds:

$$b=\frac{a+c+\sqrt{a^2+10\cdot a\cdot c+c^2}}{4}. \tag{46}$$

If a is fixed, since a<b<c, from equality (46) we obtain that:

$$a<\frac{a+c+\sqrt{a^2+10\cdot a\cdot c+c^2}}{4}<c. \tag{56}$$

$$c\in(2\cdot a,3\cdot a) \quad \text{and} \quad b\in\left(2\cdot a,\frac{(2+\sqrt{10})}{2}\cdot a\right). \tag{57}$$

Equalities (1), (2), (5), (6) and (7) are also valid in the case of: a<b<c. But, moreover, in this case, none of the equalities from (7′) hold; so, in this case, the equality θ=90°, it is impossible. What's more, not even equality θ=120°, it is not possible in this case.

Case 2: b<c<a. In this case, see Figure 8, below.

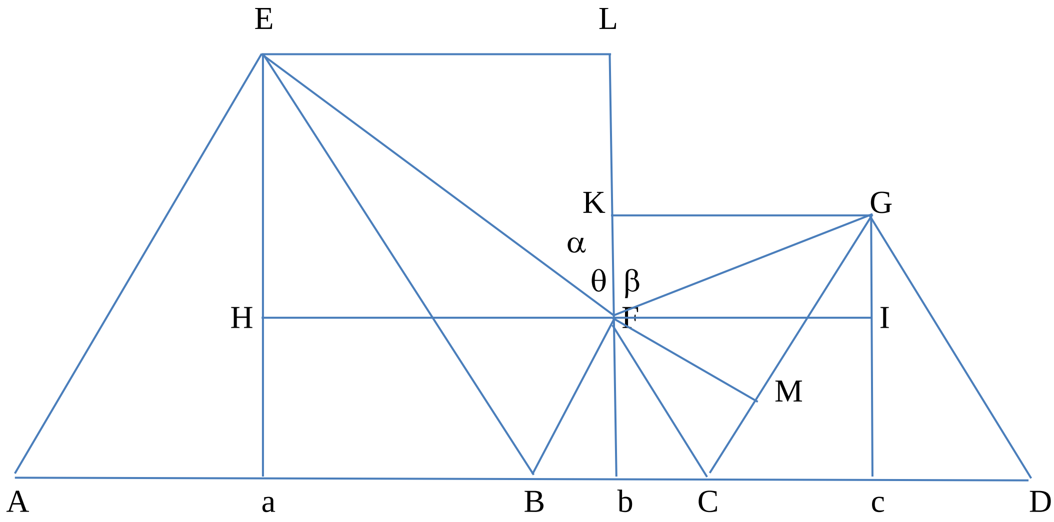

Figura 8.

Equalities (1), (2), (4), (5), (6), (6′) and (7) also hold in this case. Moreover, also the first equality from (7′) holds. So in this case,

$\theta=90°$ exactly if: $b=a+c-\sqrt{a^2+a\cdot c+c^2}$. (58)

Next we will prove that the following equivalence holds:

$\theta=120°$ is equivalent to $b=\frac{a\cdot c}{a+c}$. (59)

Indeed, from equality (13) and equivalence (26), it follows that:

$\theta=120°$ if and only if $b^2\cdot(a+c)-b\cdot(a^2+3\cdot a\cdot c+c^2)+(a^2\cdot c+a\cdot c^2)=0$.(60)

Since, b<c<a, it follows that the equivalence (59) holds.

***Remark 11*:** Let be GI ⊥ CD, LF ⊥ BC, EH ⊥ AB, GK ⊥ GI and EL ⊥ LF. Then:

$EL=FH=\frac{a+b}{2}$, $KG=FI=\frac{b+c}{2}$,

$FK=GI=\frac{\sqrt{3}}{2}\cdot(c-b)$ and $EH=LF=\frac{\sqrt{3}}{2}\cdot(a-b)$. (61)

If we note m(∢EFG)=θ, m(∢EFL)=α and m(∢LFG)=β, then,

$\alpha+\beta=\theta$ and $tg\theta=tg(\alpha+\beta)=\frac{tg\alpha+tg\beta}{1-tg\alpha\cdot tg\beta}$. (62)

But,

$tg\alpha=\frac{EL}{FL}=\frac{a+b}{\sqrt{3}\cdot(a-b)}$ and $tg\beta=\frac{GK}{FK}=\frac{c+b}{\sqrt{3}\cdot(c-b)}$. (63)

Now from equalities (62) and (63) we get the equality from (59).

***Remark 12*:** Let be FM ⊥ CG. Then:

$CM=\frac{b}{2}$, $FM=\frac{b\cdot\sqrt{3}}{2}$, $EH=\frac{\sqrt{3}\cdot(a-b)}{2}$ and $GM=c-\frac{b}{2}$. (64)

From equalities (64), we obtain that the equivalence holds:

$b=\frac{a\cdot c}{a+c}$ is equivalent to: $\frac{GM}{HF}=\frac{FM}{HE}$.

So,

$b=\frac{a\cdot c}{a+c}$ is equivalent to: ΔEHF ≅ ΔFMG. (65)

From this relationship of similarity, it follows that m(∢EFH)=m(∢FGM) and thus, θ=120°.

***Remark 13*:** We extend the segment [AE] with the segment [EN] and we extend the segment [DG] with the segment [GP], so that:

EN=c and GP=a. (66)

See Figure 9. Then ΔBEN ≡ ΔPGC (LAL) and as a consequence:

$m(\sphericalangle ENB)=m(\sphericalangle GCP) \overset{not.}{=} \varphi$ and $m(\sphericalangle EBN)=m(\sphericalangle GPC) \overset{not.}{=} \psi$.

Well, more:

$\varphi+\psi=60°$, because $m(\sphericalangle BEN)=m(\sphericalangle CGP)=120°$.

Therefore,

$b=\frac{a\cdot c}{a+c}$ is equivalent to: $\frac{a+c}{a}=\frac{c}{b}$,

that is:

ΔABN ≅ ΔCGF. (67)

From this relationship of similarity, it follows that:

$m(\sphericalangle CFG)=m(\sphericalangle ANB) \overset{not.}{=} y=\psi+60°$. (68)

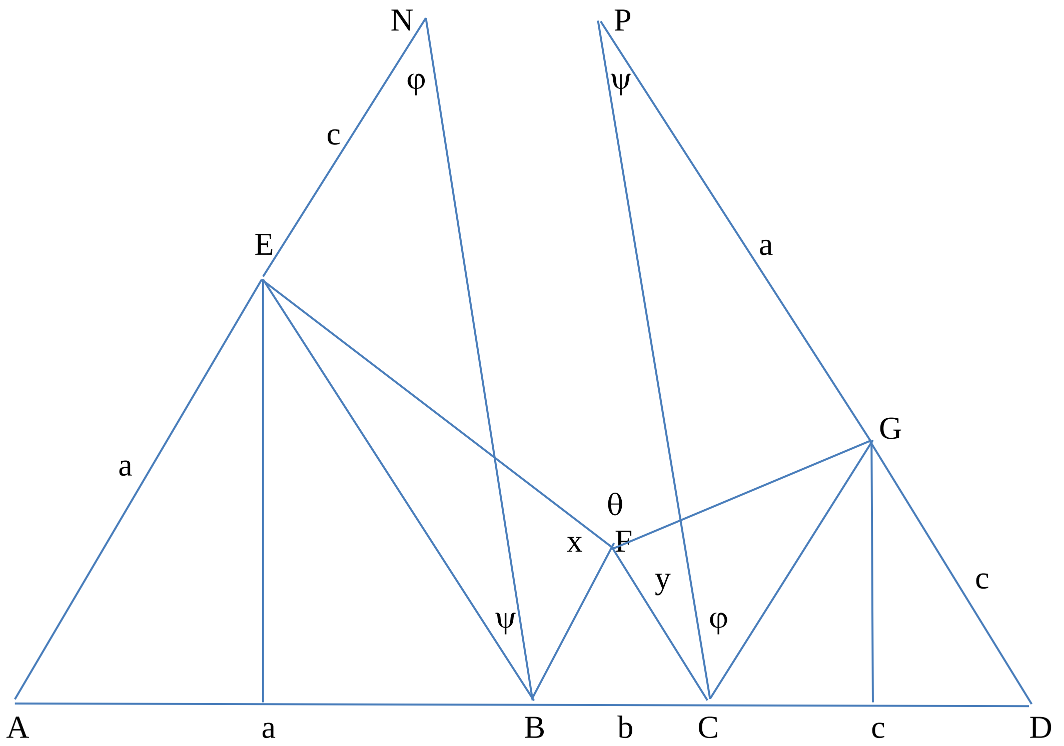

Figure 9.

Analogously we obtain:

$b=\frac{a\cdot c}{a+c}$ is equivalent to: $\frac{a+c}{c}=\frac{a}{b}$,

that is:

ΔFBE ≅ ΔCDP. (69)

From this relationship of similarity, it follows that:

$m(\sphericalangle BFE)=m(\sphericalangle DCP) \overset{not.}{=} x=\varphi+60°$. (70)

From the hypothesis and the equalities (68) and (70), it follows that θ=120°.

If θ=135°, then from the equalities (37′), (38), (39), (40) and (41), it follows that, in this case, only the equality (40) and the second equality from (41) hold. So in this case,

$$(\sqrt{3}-1)\cdot b^2+2\cdot b\cdot(a+c)-a\cdot c\cdot(\sqrt{3}+1)=0, \quad (40)$$

and thus:

$$b=\frac{\sqrt{3}+1}{2}\cdot\left(-a-c+\sqrt{a^2+4\cdot a\cdot c+c^2}\right) \quad \text{and} \quad \frac{\sqrt{3}+1}{2}\cdot\left(-a-c+\sqrt{a^2+4\cdot a\cdot c+c^2}\right)<c<a. \qquad (71)$$

From the inequalities in (71), we obtain that if c is fixed, then:

$$a\in\left(c,(2+\sqrt{3})\cdot c\right) \quad \text{and} \quad b\in\left(\frac{(1+\sqrt{3})\cdot(\sqrt{6}-2)}{2}\cdot c, c\right). \qquad (72)$$

If θ=150°, then equality (37′) holds:

$$[2\cdot b^2-b\cdot(a+c)-a\cdot c]\cdot[b^2+b\cdot(a+c)-2\cdot a\cdot c]=0. \qquad (45')$$

Since b<a<c, it follows that the equalities from (46):

$$b=\frac{a+c-\sqrt{a^2+10\cdot a\cdot c+c^2}}{4}<0 \quad \text{and} \quad b=\frac{a+c+\sqrt{a^2+10\cdot a\cdot c+c^2}}{4}, \qquad (46)$$

cannot take place. So in this case, we have the equation:

$$b^2+b\cdot(a+c)-2\cdot a\cdot c=0, \qquad (73)$$

with the solutions:

$$b=\frac{-a-c-\sqrt{a^2+10\cdot a\cdot c+c^2}}{4}<0 \quad \text{and} \quad b=\frac{-a-c+\sqrt{a^2+10\cdot a\cdot c+c^2}}{4}. \qquad (74)$$

Since the first equality from (46) cannot occur, and b<c<a, from the second equality from (74), it follows that:

$$c<a<\frac{-a-c+\sqrt{a^2+10\cdot a\cdot c+c^2}}{4}. \qquad (75)$$

From the inequalities in (75), we obtain that if c is fixed, then:

$$a\in\left(\frac{c}{2},2\cdot c\right) \quad \text{and} \quad b\in\left(\frac{c}{2},c\right). \qquad (76)$$

***Otherwise*:** We use Figure 9 and equations (62) and (63). Thus, the equalities:

$$tg\theta=tg(\alpha+\beta)=tg(150°)=-\frac{1}{\sqrt{3}},$$

are equivalent to equality (71).

***Remark 14*:** For reasons of symmetry, we also obtain the results from Case 2 in case b<a<c.

<u>Case 3:</u> a<b<c, as in Figure 10.

In this case too, the equalities (1), (2), (6′), (11), (13′), (37′) and (45′) hold. Therefore,

➢ θ=90° if and only if $b^2-2\cdot b\cdot(a+c)+a\cdot c=0$. (7)

Since no equality from (7′) is possible in this case, it follows that the equivalence (7) is not possible.

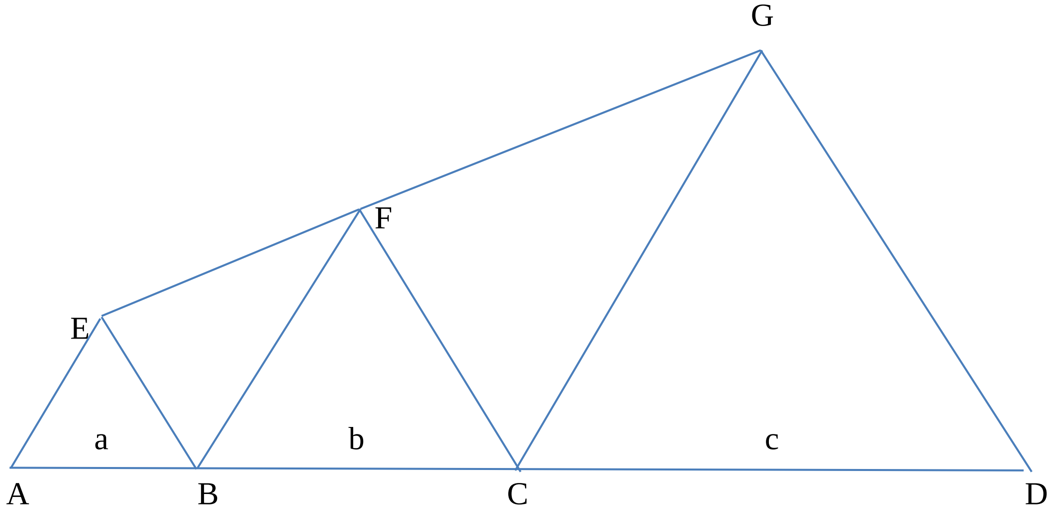

Figure 10.

- $\theta=120°$ if and only if $b^2\cdot(a+c)-b\cdot(a^2+3\cdot a\cdot c+c^2)+(a^2\cdot c+a\cdot c^2)=0.$ (13′)

Since no equality from (14) is possible in this case, it follows that the equivalence (13′) is not possible.

- $\theta=135°$ if and only if

$[(\sqrt{3}+1)\cdot b^2-2\cdot b\cdot(a+c)-a\cdot c\cdot(\sqrt{3}-1)]\cdot[(\sqrt{3}-1)\cdot b^2+2\cdot b\cdot(a+c)-a\cdot c\cdot(\sqrt{3}+1)]=0.$ (37′)

Equation (37′) admits two convenient solutions for this case:

$b=\frac{\sqrt{3}-1}{2}\cdot(a+c+\sqrt{a^2+4\cdot a\cdot c+c^2}),$

in which case:

$c>(2+\sqrt{3})\cdot a$ and $b>(2+\sqrt{3})\cdot a$ (77)

and respectively:

$b=\frac{\sqrt{3}-1}{2}\cdot(a+c+\sqrt{a^2+4\cdot a\cdot c+c^2}),$

in which case:

$c>(2+\sqrt{3})\cdot a$ and $b>(2+\sqrt{3})\cdot a$ (78)

$b=\frac{\sqrt{3}+1}{2}\cdot(-a-c+\sqrt{a^2+4\cdot a\cdot c+c^2}),$

in which case:

$c>(2+\sqrt{3})\cdot a$ and $b>a.$ (79)

- $\theta=150°$ if and only if

$[2\cdot b^2-b\cdot(a+c)-a\cdot c]\cdot[b^2+b\cdot(a+c)-2\cdot a\cdot c]=0.$ (45′)

Equation (45′) admits two convenient solutions for this case:

$b=\frac{1}{4}\cdot(a+c+\sqrt{a^2+10\cdot a\cdot c+c^2}),$

in which case:

$c>2\cdot a$ and $b>2\cdot a$ (80)

and respectively:

$b=\frac{1}{2}\cdot(-a-c+\sqrt{a^2+4\cdot a\cdot c+c^2}),$

in which case:

$c>2\cdot a$ and $b>a.$ (81)

- $\theta=180°$ if and only if

$4\cdot(a^2-a\cdot b+b^2)\cdot(b^2-b\cdot c+c^2)=[b^2-2\cdot b\cdot(a+c)+a\cdot c]^2.$ (6″)

Equation (6″) is equivalent to:

$(b^2-a\cdot c)^2=0$ (6‴)

and admits the convenient solution for this case:

$b=\sqrt{a\cdot c}.$ (82)

***Otherwise*:** We are using Figure 11, below, where:

$FI\perp BC,$ $GJ\perp CD,$ $EK\perp FI,$ and $FL\perp GJ.$

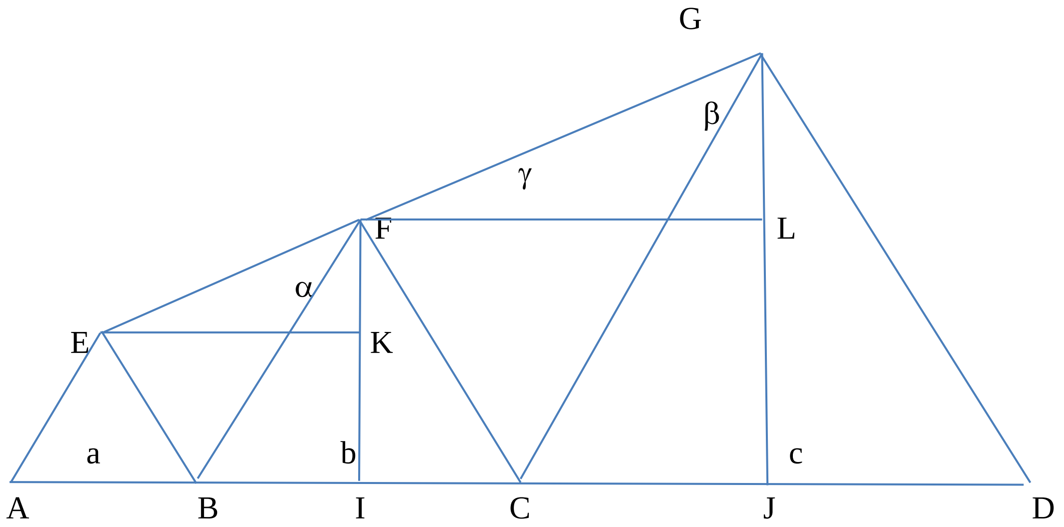

Figure 11.

If ∢EFK=α and ∢FGL=β, then:

$$tg\alpha=\frac{FK}{EK}=\frac{a+b}{\sqrt{3}\cdot(b-a)} \quad \text{and} \quad tg\beta=\frac{FL}{GL}=\frac{c+b}{\sqrt{3}\cdot(c-b)}. \tag{83}$$

If:

$$b=\sqrt{a\cdot c}, \tag{82}$$

then, from the equalities (82) and (83), it follows that:

$$tg\alpha=\frac{\sqrt{a}+\sqrt{c}}{\sqrt{3}\cdot(\sqrt{c}-\sqrt{a})}=tg\beta, \quad \text{i.e.} \quad \alpha=\beta. \tag{83}$$

Reciprocally, if:

$$tg\alpha=tg\beta, \tag{84}$$

then, from equalities (83) and (84), it follows that equality (82) holds. But, the equalities from (83), are equivalent to the fact that:

$$\Delta EKF \cong \Delta FLG. \tag{85}$$

But, the similarity in (85) is equivalent to the fact that ∢FEK ≡ ∢GFL and because m(∢KFL)=90°, it follows that:

$$\alpha+\beta=90°, \quad \text{i.e.} \quad \theta=180°.$$

Therefore,

$$F\in EG \quad \text{exactly if} \quad b=\sqrt{a\cdot c}. \tag{86}$$

***Remark 15*:** We assume that the points E, F and G are collinear and the line EG intersects the line AD at a point, which we denote by U. If AU=x, then, from the similarity of the triangles UBE, UCF and UDG, it follows that:

$$\frac{x}{x+a}=\frac{a}{b} \quad \text{and} \quad \frac{x+a}{x+a+b}=\frac{b}{c}. \tag{87}$$

From the equalities in (87) it follows that:

$$x=\frac{a^2}{b-a}, \quad \text{respectively} \quad x=\frac{b^2+a\cdot b-a\cdot c}{c-b}. \tag{88}$$

Now, from the equalities in (88), the equality (82) follows.

***Remark 16*:** According to the above notations and equalities (83), if ∢GFL=γ, then:

$$tg\alpha=\frac{FK}{EK}=\frac{a+b}{\sqrt{3}\cdot(b-a)} \quad \text{and} \quad tg\gamma=\frac{GL}{FL}=\frac{\sqrt{3}\cdot(c-b)}{c+b}. \tag{89}$$

Now, from the above, we observe that the equality tgα·tgγ=1 is equivalent to equality (82).

Remark 17: For reasons of symmetry, we obtain the results from Case 3 also in the case $c<b<a$.

We conclude this paragraph with the following remark:

Remark 18: If the equilateral triangles $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$, ..., $P_1P_2P_3$, $Q_1Q_2Q_3$, $R_1R_2R_3$, of sides, respectively a, b, c, ..., p, q, r, are located so that the points A_1, A_2, B_1, B_2, C_1, C_2, ..., P_1, P_2, Q_1, Q_2, R_1, R_2, are collinear on the line d, and the points A_3, B_3, C_3, ... , P_3, Q_3, R_3, are located on the same side of the line d, then:

A_3, B_3, C_3, ... , P_3, Q_3, R_3, are collinear exactly if $b\cdot c=p\cdot q$. (90)

Equality (82) is equivalent to the following equality:

$S^2_{\Delta ABE}=S_{\Delta BCF}\cdot S_{\Delta CDG}$; (90)

also, the equality from (90) is equivalent to:

$$S_{\Delta B_1B_2B_3}\cdot S_{\Delta C_1C_2C_3}=S_{\Delta P_1P_2P_3}\cdot S_{\Delta Q_1Q_2Q_3}. \quad (91)$$

3. Conclusions And Recommendations

So, by "*playing*" with three equilateral triangles of different sides, but each triangle having one of the sides on the same line d and the other sides on the same side of the line d, we can get nice and interesting results. All the results presented in Paragraph 2 can be translated into problems and presented to students of certain ages and abilities.

We propose to the reader attentive and interested in these issues, the following problems to be solved:

1) Determine the relations between the numbers a, b, and c, if $\theta\in\{15°,30°,45°,60°,75°,105°,165°\}$.

2) Determine the relationships between the areas of the triangles that appear in the figures above.

3) Determine the angle θ, if a=c. Prove that in this case, $\theta>60°$.

Hint: It is shown that:

$\cos\theta=\dfrac{a^2-4\cdot a\cdot b+b^2}{2\cdot(a^2-a\cdot b+b^2)}$, or equivalent: $\sin\dfrac{\theta}{2}=\dfrac{a+b}{2\cdot\sqrt{a^2-a\cdot b+b^2}}$.

4) Study / determine all the metric and areolar relations that occur in the situation where the triangle of side b is located on the other side of the line d.

At the end of this paper, we make the statement that we are convinced that we have given the reader a good picture of how students' creativity can be formed and developed by studying / solving problems of this type

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